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ON THE NUMBER OF SOLUTIONS TO A CLASS OF COMPLEMENTARITY PROBLE--ETC(U)
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M. Kojima and R. Saigal

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ABSTRACT

In this paper ~~we consider~~ the problem of establishing the number of solutions to the complementarity problem is considered. For the case when the Jacobian of the mapping has all principal minors negative, and statisfies a condition at infinity, ~~we prove~~ it is proved that the problem has either 0,1,2 or 3 solutions. ~~We also show~~ It is shown that when the Jacobian has all principal minors positive, and satisfies a condition at infinity, the problem has a unique solution. → to p. -B-

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Significance and Explanation

cont

The problem of solving nonlinear programs, certain nonlinear n person noncooperative games, general equilibrium models with linear production and several others can be stated as a complementarity problem on a closed convex and polyhedral cone. In this paper we consider the problem of establishing the number of solutions such problems may have. The basic tool used is the homotopy invariance of the Brouwer degree and the theorems of Gale and Nikaido, and Inada. For the case when this cone is the non-negative orthant, the underlying functions are continuously differentiable and satisfy a condition at infinity, and the Jacobian has either all principal minors positive or negative, the exact number of solutions of the problem are obtained. It is shown that for the positive case, this number is one, and for the negative case, it can be either 0,1,2, or 3. In the negative case, conditions when the problem has a unique solution are also given. These results can have important applications in general equilibrium analysis. In addition, when the problem is defined on a closed, convex, polyhedral and pointed cone, with a positivity condition on the Jacobian (similar to the one put by Mas-Colell in a recent extension of the Gale-Nikaido theorem) using the method of Kojima and Saigal, a uniqueness result is established. Such complementarity problems arise in the general equilibrium models with linear production, and have been recently considered by Kehoe.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

ON THE NUMBER OF SOLUTIONS

TO A CLASS OF COMPLEMENTARITY PROBLEMS

M. Kojima and R. Saigal

§1 Introduction:

Let R^n be the n-dimensional Euclidean space, $K \subseteq R^n$ be a closed, convex and polyhedral cone that is pointed (i.e., $K \cap -K = \{0\}$) and R^n_+ be the subset of all non-negative vectors in R^n. Given a mapping $f: K \to R^n$, and an n-vector q in R^n, in this note we consider the problem of establishing the number of solutions to the problem of finding an x such that

$$x \in K, \quad f(x) - q \in K^+, \quad \langle x, f(x) - q \rangle = 0 \tag{1.1}$$

where K^+ is the polar cone of K, i.e., $K^+ = \{y: \langle x,y \rangle \geq 0 \text{ for all } x \text{ in } K\}$. In case $K = R^n_+$, this problem is called the non-linear complementarity problem, and has been considered by several authors. A partial list of these include Cottle [1], Karamardian [4], Megiddo and Kojima [11], Saigal and Simon [16].

Our aim in this paper is to make some statement about the solution set of (1.1) for all q in R^n. For the special case when f is affine, and $K = R^n_+$, two such results exist, namely those of Kojima and Saigal [8] when the Jacobian of f has all principal minors negative and of Samelson, Thrall and Wesler [17] and Murty [12] when the Jacobian of f has all principal minors positive. For the nonlinear case considered in this paper, we will assume

(1.2) $\quad f(0) = 0$ and f is continuously differentiable.

And when $Df(x)$ has all principal minors negative for each x in $K = R^n_+$, we will reproduce the main result of Kojima and Saigal [8] that for any q (1.1) has 0,1,2, or 3 solutions. This will be established with f satisfying the additional assumption

(1.3) $\quad$ for any sequence $\{x_k\}_{k=1}^{\infty}$ in R^n_+ such that $\|x_k\| \to \infty$, there exists a subsequence J such that either there is an i such that $f_i(x^k) \to -\infty$ for k in J or there is an i such that $x_i^k > 0$ for all k in J and $f_i(x^k) \to \infty$ for k in J.

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In addition, using the method of Kojima and Saigal [7] (see also Mas-Collel [10]) when f satisfies (1.2) and K is an arbitrary cone, with an appropriate condition on the Jacobian $Df(x)$ we will show that for each q (1.1) has a unique solution. When $K = R^n_+$, this condition reduces to the fact that $Df(x)$ is a P-matrix (i.e., has all principal minors positive), and f satisfies some condition at infinity.

The principal tool used in the proof of the above mentioned results is degree theory. We will follow the notation of Ortega and Rheinboldt [14] for this purpose. As suggested by Megiddo and Kojima [11], to facilitate the use of this theory, we now formulate (1.1) as an equation solving problem.

Define the projection mapping $P: R^n \to K$ by

$$\|P(x) - x\| = \min_{y \in K} \|y - x\|$$

and the vectors

$$x^+ = P(x) \in K$$

and

$$x^- = x - P(x) \in -K^+$$

In case $K = R^n_+$, the above operation simplifies to

$$x_i^+ = \max\{0, x_i\}$$

$$x_i^- = \min\{0, x_i\} \quad .$$

Now, define the mapping $g: R^n \to R^n$ by

$$g(x) = x^- + f(x^+)$$

and for some q in R^n, the problem of solving systems of equations

$$g(x) = q \quad . \tag{1.4}$$

It can be readily confirmed that for a given q in R^n, if x solves (1.1), then $x^+ = x$, $x^- = -f(x) + q$ and $z = x^- + x^+$ solves (1.4). Also, if x solves (1.4), then x^+ solves (1.1).

In section 2, for the case when f satisfies (1.2) and (1.3), we establish some properties of the mapping g when $K = R^n_+$. In section 3 we establish the main result relating to the negative principal minors, in section 4 we prove a sufficient condition for uniqueness when K is a convex, polyhedral and pointed cone, the Jacobian of f satisfies certain positivity conditions, and that g is norm coercive. Finally in the appendix we prove a PL homeomorphism theorem. The results of section 3 can be extended to an arbitrary cone if a generalization of the theorem of Inada [3] similar to the extension of the Gale-Nikaido theorem [2] proved by Mas-Colell [10] can be established.

§2. Some properties of the mapping g :

In this section, when $K=R^n_+$, we establish some important properties of the mapping g as defined in section 1. Let $N=\{1,2,\ldots,n\}$ and for each $I \subseteq N$, possibly empty, define

$$\sigma(I) = \{x \in R^n : x_i \geq 0 \text{ for } i \in I,\ x_i \leq 0 \text{ for } i \notin I\}$$

and $\Sigma = \{\sigma(I) : I \subseteq N\}$. We note the Σ subdivides R^n, i.e., (R^n,Σ) is a subdivided polyhedron. In addition, we note that since f is continuously differentiable, for each $I \subseteq N$, $g_I \equiv g|\sigma(I) : \sigma(I) \to R^n$ is continuously differentiable, and if $A \equiv Dg_I(x)$ for some x in $\sigma(I)$, the j^{th} row A_j of A is u_j (the j^{th} unit vector) for $j \notin I$ and $Df(x^+)_j$ (the j^{th} row of $Df(x^+)$) for $j \in I$.

We now establish a

<u>Theorem 2.1</u>. Let f satisfy (1.2) and (1.3), and let g be defined as in section 1. Then g is norm coercive, i.e., if $\{x^k\}$ is a sequence such that $\|x^k\| \to \infty$ then $\|g(x^k)\| \to \infty$.

<u>Proof</u>: For the sequence $\{x^k\}$ define $y^k = (x^k)^-$ and $z^k = (x^k)^+$. Since g is continuous, it is clearly norm coercive if the sequence $\{z^k\}$ is bounded. Thus assume $\|z^k\| \to \infty$. From condition (1.3), there is a subsequence J and an i such that $z_i^k > 0$ and $|f_i(z^k)| \to +\infty$ for all k in J. Since $x_i^k = z_i^k$, we have $g_i(x^k) = f_i(z^k)$, and we have our result. Also, if $f_i(z^k) \to -\infty$ as $z^k \to \infty$, then since $y_i^k \leq 0$, $g_i(x^k) \leq f_i(z^k)$ thus $|g_i(x^k)| \to +\infty$ and the theorem follows.

<u>Theorem 2.2</u>. Let f satisfy (1.2) and for each x, $Df(x)$ have all principal minors negative. Then for each $I \subseteq N$, $g_I : \sigma(I) \to R^n$ is one to one.

<u>Proof</u>: The theorem holds trivially for $I = \emptyset$ (the empty set), since then $g_I = \text{id}$ (the identity map). Now, let $\emptyset \neq I \subseteq N$, and let $g(x_1) = g(x_2)$ for some x_1, x_2 in $\sigma(I)$. Hence

$$x_1^- + f(x_1^+) = x_2^- + f(x_2^+) \tag{2.1}$$

and

$$f_i(x_1^+) = f_i(x_2^+) \text{ for all } i \in I \tag{2.2}$$

Define the mapping $f^I : R_+^{|I|} \to R^{|I|}$ as follows:

let the elements in I be $i_1 < i_2 < \ldots < i_{|I|}$. Then, define a $n \times |I|$ matrix P whose j^{th} column is u_{i_j} (the i_j^{th} unit vector). $f^I \equiv P^T f P$ (the diagram below may help in understanding f^I):

$$\begin{tikzcd} R_+^n \arrow[r, "f"] & R^n \\ R_+^{|I|} \arrow[u, "P"] \arrow[r, "f^I"'] & R^{|I|} \arrow[u, "P"'] \end{tikzcd} .$$

Since f is differentiable, so is f^I and $Df^I(x) = P^T Df(Px) P$ and is thus a principal minor of $Df(x)$. Since $Df(x)$ has all principal minors negative, so does $Df^I(x)$. Also, because of (2.2) $f^I(P^T x_1^+) = f^I(P^T x_2^+)$. Now, using the well known theorem of Inada [3] (see also Theorem 20.4, Nikaido [13]) on a cubical region containing $P^T x_1^+$ and $P^T x_2^+$ we conclude that

$$P^T x_1^+ = P^T x_2^+ .$$

Also, since x_1 and x_2 are in $\sigma(I)$, $x_1^+ = x_2^+$. Thus, from (2.1) $x_1^- = x_2^-$ and we have our result.

We now establish a set of sufficient conditions under which g is locally univalent.

Theorem 2.3 (local univalence theorem). Let $x \in R^n$, $Df(x)$ has all principal minors negative, and $x \nleq 0$. Then, there is an open neighborhood U of x which g maps U homomorphically onto $g(U)$.

Proof: Let x belong to the pieces $\sigma(I_1), \sigma(I_2), \ldots, \sigma(I_k)$. Since $x \nleq 0$, $I_j \neq \emptyset$ for each $j = 1,\ldots,k$. Thus $\det Dg_{I_j}(x) < 0$ for each $j = 1,\ldots,k$. The result now follows from Theorem A1 in the appendix and Lemma 2.11 of Kojima [6].

§3. The negative case:

In this section we consider the problem (1.1) with $K = R^n_+$, f satisfies (1.2)-(1.3) and $Df(x)$ has all principal minors negative. We will prove that for any q, (1.1) has 0,1,2 or 3 solutions. Before we prove our main result, we now establish the degree of the mapping g.

3.1 Calculation of the degree of g:

Following Ortega and Rheinboldt [14], given a continuous mapping g on an open set U, and a $y \notin g(\partial U)$, where ∂U is the boundary of the set U we denote by $\deg(g,U,y)$ the degree of g with respect to U at y. Now, for a given $n \times n$ matrix M which has all principal minors non-zero, define the piecewise linear mapping

$$\ell(x) = x^- + Mx^+ \tag{3.1}$$

on the subdivided polyhedron (R^n, Σ). Since M has all principal minors non-zero, each linear mapping $\ell_I \equiv \ell|\sigma(I)$ is one to one. Also, $\ell(x)$ is norm coercive. We can then prove:

Theorem 3.1.

(a) Let M have all principal minors negative, and $M \nless 0$. Then $\deg(\ell,R^n,q) = -1$ for all q in R^n.

(b) Let M have all principal minors negative, and $M < 0$. Then $\deg(\ell,R^n,q) = 0$ for all q in R^n.

Proof. Let $q^* > 0$ be such that $S(q^*) = \{x : \ell(x) = q^*\} \cap \partial\sigma(I) = \emptyset$ for every $I \subseteq N$. From Theorem 3.1, Kojima and Saigal [8], under hypothesis (a), $\# S(q^*) = 1$. Hence $\deg(\ell,R^n,q^*) = -1$; and from Lemma 2.2, [8], under hypothesis (b), $\deg(\ell,R^n,q^*) = 0$. Now, let $q \in R^n$, and consider the homotopy; for t in $[0,1]$,

$$L(x,t) = \ell(x) - (1-t)q^* - tq .$$

Since ℓ is norm coercive, $L^{-1}(0)$ is bounded, and thus from the homotopy invariance theorem [6.2.2, 14] $\deg(\ell,R^n,q) = \deg(\ell,R^n,q^*)$ and the theorem follows.

Now, let $M = Df(0)$. Since f satisfies (1.2), $f(x) = Mx + o(x)$ such that $\|o(x)\| \to 0$ as $\|x\| \to 0$. Now, for t in $[0,1]$ define the homotopy

$$H(x,t) = (1-t)\,\ell(x) + t\,g(x)$$
$$= \ell(x) + t\,o(x)\,. \tag{3.2}$$

Lemma 3.2. Let M have all principal minors non-zero. Then there exist $\alpha > 0$, and $\varepsilon > 0$ such that $\|H(x,t)\| \geq \alpha\varepsilon$ for all x in the boundary $\partial B(\varepsilon)$ of $B(\varepsilon) = \{x : \|x\| < \varepsilon\}$.

Proof: Since M has all principal minors non-zero, the Jacobians of the linear mappings $\ell_I \equiv \ell|\sigma(I)$ are non-singular. Thus, there exists an $\alpha > 0$ such that $\|\ell(x)\| \geq 2\alpha\|x\|$ for all x in R^n. Now, as $\|o(x)\| \to 0$ as $\|x\| \to 0$, there exists an $\varepsilon > 0$, sufficiently small, such that $\|o(x)\| < \alpha\varepsilon$ for all x in $B(\varepsilon)$. Thus, from (3.2), $\|H(x,t)\| \geq \|\ell(x)\| - \|o(x)\| \geq 2\alpha\|x\| - \alpha\varepsilon$, and , for $x \in \partial B(\varepsilon)$, $\|H(x,t)\| \geq \alpha\varepsilon$.

Lemma 3.3. Let f satisfy (1.2), (1.3) and let $Df(x)$ have each principal minor negative for all x. Then, for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for all q with $\|q\| < \delta$, $S(q) = \{x : g(x) = q\} \subset B(\varepsilon)$.

Proof: Assume the contrary. Then, for some $\varepsilon > 0$ and every $\delta > 0$ there exists a q such that $\|q\| \leq \delta$ and $S(q) \not\subset B(\varepsilon)$. Choose $\delta_k \to 0$ and let $x_k \in S(q_k)$ be such that $x_k \notin B(\varepsilon)$. Now $\{x_k\}$ is bounded, since otherwise, from Theorem 2.1, $\|q_k\| \to \infty$. Thus, on some subsequence $x_k \to x_\infty$. Also $\|x_k\| \geq \varepsilon$ thus $\|x_\infty\| \geq \varepsilon$. But $g(x_k) = q_k \to 0$. Hence $g(x_\infty) = 0$. Let $x_\infty \in \sigma(I)$. Since $g(0) = 0$, and $0 \in \sigma(I)$ we contradict the conclusion of Theorem 2.2.

We now prove the main theorem, which is also a nonlinear version of Theorem 3.1.

Theorem 3.4. Let f satisfy (1.2) and (1.3) and let g be defined as in section 1.

(a) Let $Df(x)$ have all principal minors negative for each x in R^n_+, and $M = Df(0) \nless 0$. Then, $\deg(g,R^n,q) = -1$ for each q in R^n.

(b) Let $Df(x)$ have all principal minors negative for each x in R^n_+, and $M = Df(0) < 0$. Then $\deg(g,R^n,q) = 0$ for each q in R^n.

Proof: Let α and ε be as in Lemma 3.2 and let $\delta > 0$ be as in lemma 3.3, and sufficiently small, so that $\delta < \alpha\varepsilon$. For $d > 0$, define $0 < \lambda < \min\{\varepsilon/\alpha\ , \delta/\|Md\|\}$ and for $q^* = \lambda Md$ the linear problem

$$\ell(x) = q^* \tag{3.3}$$

Under the hypothesis (a), there is a $d > 0$ such that $Md > 0$. Define $\lambda > 0$ as above and the linear problem (3.3) for $q^* = \lambda Md$. Using theorem 3.1 [8], this problem has the unique solution $x = \lambda d \in B(\varepsilon)$. Now consider the homotopy (3.2). $H^{-1}(q^*) \cap \partial B(\varepsilon) = \emptyset$, since from Lemma 3.2 and the choice of δ, for $x \in \partial B(\varepsilon)$, $\|H(x,t) - q^*\| \geqq \|H(x,t)\| - \|q^*\| \geqq \alpha\varepsilon - \delta > 0$. Thus, using the homotopy invariance theorem, [6.2.2, 14] $\deg(g,R^n,q^*) = \deg(g,B(\varepsilon),q^*) = \deg(\ell,B(\varepsilon),q^*) = -1$.

Under the hypothesis (b), let $d > 0$ and $0 < \lambda < \min\{\varepsilon,\delta\}/\|d\|$. Then, the linear problem for $q^* = \lambda d$ has no solution, Lemma 2.2 [8]. Using the arguments as above, we can establish that

$$\deg(g,R^n,q^*) = \deg(g,B(\varepsilon),q^*) = \deg(\ell,B(\varepsilon),q^*) = 0 .$$

Now, for q in R^n, consider the homotopy: for t in $[0,1]$

$$L(x,t) = g(x) + (1-t)q^* + tq .$$

From Theorem 2.1, since $g(x)$ is norm coercive, $L^{-1}(0)$ is bounded. Using the homotopy invariance theorem [6.2.2, 14], $\deg(g,R^n,q) = \deg(g,R^n,q^*)$, the theorem follows.

3.2. <u>The Number of solutions</u>:

We now establish the required results on the number of solutions to (1.1) for any given q. We now establish a simple lemma:

<u>Lemma 3.4</u>. For some q, let x solve (1.4), and let $Df(x)$ have all principal minors non-zero. Then there is an open neighborhood U of x such that x is the only solution to (1.4) in U.

<u>Proof</u>: The proof follows from the fact that, under our hypothesis, $Dg_I(x)$ is non-singular for each I such that x is in $\sigma(I)$, and thus, by the inverse function theorem, [5.2.1, 14], x is the only solution of $g_I(x) = q$ in a small neighborhood $U(I)$. See also Corollary 4.7, Mangasarian [9].

We are now ready to establish our main results:

<u>Theorem 3.5</u>. For each x let $Df(x)$ have all principal minors negative. Then, if

(i) $Df(0) \nleq 0$, (1.1) has a unique solution for each $q \ngeq 0$ and $q = 0$, three solutions for $q < 0$ and at most two solutions for $0 \neq q \leqq 0$.

$q_i = 0$ for at least one i .

(ii) $Df(0) < 0$, (1.1) has no solution for $q \nleq 0$, one solution for $q \leq 0$, $q_i = 0$ for at least one i , and two solutions for $q < 0$.

Proof: From theorem 3.4 for every q in R^n , under hypothesis (i) $\deg(g,R^n,q) = -1$ and under (ii), $\deg(g,R^n,q) = 0$. Also, let $S(q)$ be the set of solutions of (1.1). Now, let $q \nleq 0$. Then $S(q) \cap \sigma(\phi) = \phi$. Hence at each x in $S(q)$, the conditions of Theorem 2.3 are satisfied, and thus g maps an open neighborhood U_x of x homeomorphically onto $g(U_x)$. From Theorem 3.3, Kojima and Saigal [7], $\deg(g,U_x,q) \leqq -1$. Using the decomposition of domain [6.2.7, 14], under hypothesis (i), $\#S(q) = 1$ and hypothesis (ii), $S(q) = \phi$

Now, let $q < 0$. Then $S(q) \cap \sigma(\phi) \neq \phi$, and using Theorem 2.2, $x^- = q$, $x^+ = 0$ is the unique solution in $\sigma(\phi)$. Since $g(x) = x$ for $x \in \sigma(\phi)$, there is a neighborhood U_x of x such that $\deg(g,U_x,q) = 1$. Also, if y is any other solution in $S(q)$, $\deg(q,U_y,q) = -1$ for some neighborhood U_y. Thus, using the decomposition of domain [6.2.7, 14], under hypothesis (i), $\#S(q) = 3$ and under hypothesis (ii), $\#S(q) = 2$. Now, let $0 \neq q \leqq 0$ with $q_i = 0$ for at least one i . Under hypothesis (ii), $q \in S(q)$, and using Theorem 2.2, $S(q) \cap \sigma(\phi) = \{q\}$. If there is any other solution x in $S(q)$, $\deg(g,U_x,q) = -1$. Thus $\deg(g,U_q,q) \geqq 1$. But arbitrarily close to q , there exist $q' \nleq 0$ which have no solution, i.e., $\deg(g,U_q,q') = 0$. this is a contradiction, since degree is locally a constant. Under hypothesis (i), assume $\#S(q) \geqq 3$. $q \in S(q)$. Let these solutions be $q,x^1,\ldots,x^k$. Since these are disjoint, there exist neighborhoods U_q, $U^1,\ldots,U^k$ of $q,x^1,\ldots,x^k$ respectivelly such that $U^1 \cap \operatorname{int} \sigma(\phi) = \ldots = U^k \cap \operatorname{int} \sigma(\phi) = \phi$. Thus $\deg(g,U^1,q) = \ldots = \deg(g,U^k,q) = -1$. But, the $\deg(g,R^n,q)$ is -1, hence $\deg(g,U_q,q) \geqq 1$. Since U_q is open, there exists a $q' \nleq 0$, sufficiently close to q , in U_q such that $\deg(g,U_q,q') \leqq 0$, i.e., it is zero if $S(q) \cap U_q = \phi$, and -1 otherwise. But, degree is locally constant; which is thus a contradiction and our theorem follows.

§4. The Positive Case:

In this section we consider the problem (1.1) with K a closed, convex, pointed, and polyhedral cone (i.e., for some $r \times n$ matrix A, such that $Ax = 0 \Rightarrow x = 0$, $K = \{x : Ax \leq 0\}$.) and that f satisfies (1.2). Now, with an appropriate condition on the Jacobian matrix $Df(x)$ at x, to be described below, we will prove that (1.1) has at most one solution for each q in R^n. In case $K = R^n_+$, this condition reduces to the fact that for all x outside some bounded region of K $Df(x)$ has all principal minors positive.

4.1. Condition on the Jacobian $Df(x)$.

For a given set F in R^n containing 0 let H_F be the subspace spanned by the set F, i.e.,

$$H_F = \{y : y = \sum_{i=1}^{r} \lambda_i x_i \ , \ \sum_{i=1}^{r} \lambda_i = 1 \ , \ x_i \in F\}$$

Now, let F be a face of K, i.e., there exists an $I_F \subseteq \{1,\ldots,r\}$ such that $F = \{x \in K : (Ax)_i = 0$ for each $i \in I_F\}$, and let H_F be the subspaced spanned by F. Also, let P_F be the projection onto this subspace, i.e.,

$$\|P_F(x) - x\| = \min_{y \in H_F} \|y-x\| \tag{4.1}$$

and note that P_F is a linear mapping. Thus

$$P_F \circ Df(x) : H_F \to H_F .$$

We now state the appropriate condition on $Df(x)$:

Condition 4.1: Let there exist an open bounded set U in K such that

$$\det P_F \circ Df(x) > 0 \quad \text{for all } x \text{ in } U \cap F$$

and for $x \in K \setminus U$,

$$\det P_F \circ Df(x) > 0 \quad \text{for every face } F \text{ of } K .$$

4.2 The PC^1 mapping g:

In this section we show that the mapping g is a piecewise continuously differentiable extension of f on certain subdivision of R^n. This subdivision is generated by the pieces of linearity of the piecewise linear projection mapping $P : R^n \to K$ defined by

$$\|P(x) - x\| = \min_{y \in K} \|y-x\| \tag{4.2}$$

Let Σ be the set of all subsets of R^n which are generated by closing $P^{-1}(\text{int } F)$ for some face F of K. It is clear that the elements of Σ are closed and convex. Also, that $g_\sigma = g|\sigma$ (g restricted to σ), σ in Σ, is continuously differentiable, with $Dg_\sigma(x) = P_F * Dg(x)$, where P_F is defined by (4.1), and σ = closure $(P^{-1}(\text{int } F))$.

For some sufficiently large integer $m > 0$, and $e = (1,\ldots,1)^T$ (the vector of all 1's in R^n), define $S(m) = \{x : Ax \leq 0, \ Ax \geq -me\}$. Since K is pointed, $S(m)$ is compact for each m. Let Σ_m be the pieces of linearity of the PL-mapping $P_m : R^n \to S(m)$ defined by

$$\|P_m(x) - x\| = \min_{y \in S(m)} \|y-x\| .$$

In addition, let $S'(m) = \{x : P(x) = P_m(x)\}$. Then, we can prove:

<u>Lemma 4.2</u>: Let $x_1, x_2, \ldots, x_k$ be an arbitrary finite set of vectors in R^n. For some sufficiently large $m > 0$, $x_i \in S'(m)$ for each $i = 1,\ldots,k$.

<u>Proof</u>: This lemma follows from the observation that as m approaches infinity, $S'(m)$ approaches R^n, and that any finite subset of R^n lies in a compact region. Also, see Figure 4.1.

4.3. <u>The Number of Solutions</u>:

Assume that f satisfies conditions (1.2) and (4.1), and choose an $m > 0$, sufficiently large, such that $U \cap \{x : Ax < -me\} = \phi$. Thus, the faces of $S(m)$ that are not faces of K do not intersect U. We will say that m is sufficiently large if the above property holds.

<u>Lemma 4.3</u>. Let $m > 0$ be sufficiently large, and $S(m)$ be the polyhedron defined in section 4.2. Then, the PC^1 extension g_m of $f|S(m)$ (f restricted to $S(m)$) is one to one.

<u>Proof</u>: It can be readily confirmed that under Condition 4.1 the faces of $S(m)$ that are subsets of faces of K satisfy [Condition 4.1, 7], and by the choice of $m > 0$, the faces of $S(m)$ that are not subsets of faces of K, do not intersect U, and are subsets of translates of faces of K, and thus satisfy (ii) of condition 4.1, and thus also satisfy [Condition 4.1, 7]. Now the lemma follows by a proof identical to that of [Theorem 4.3, 7].

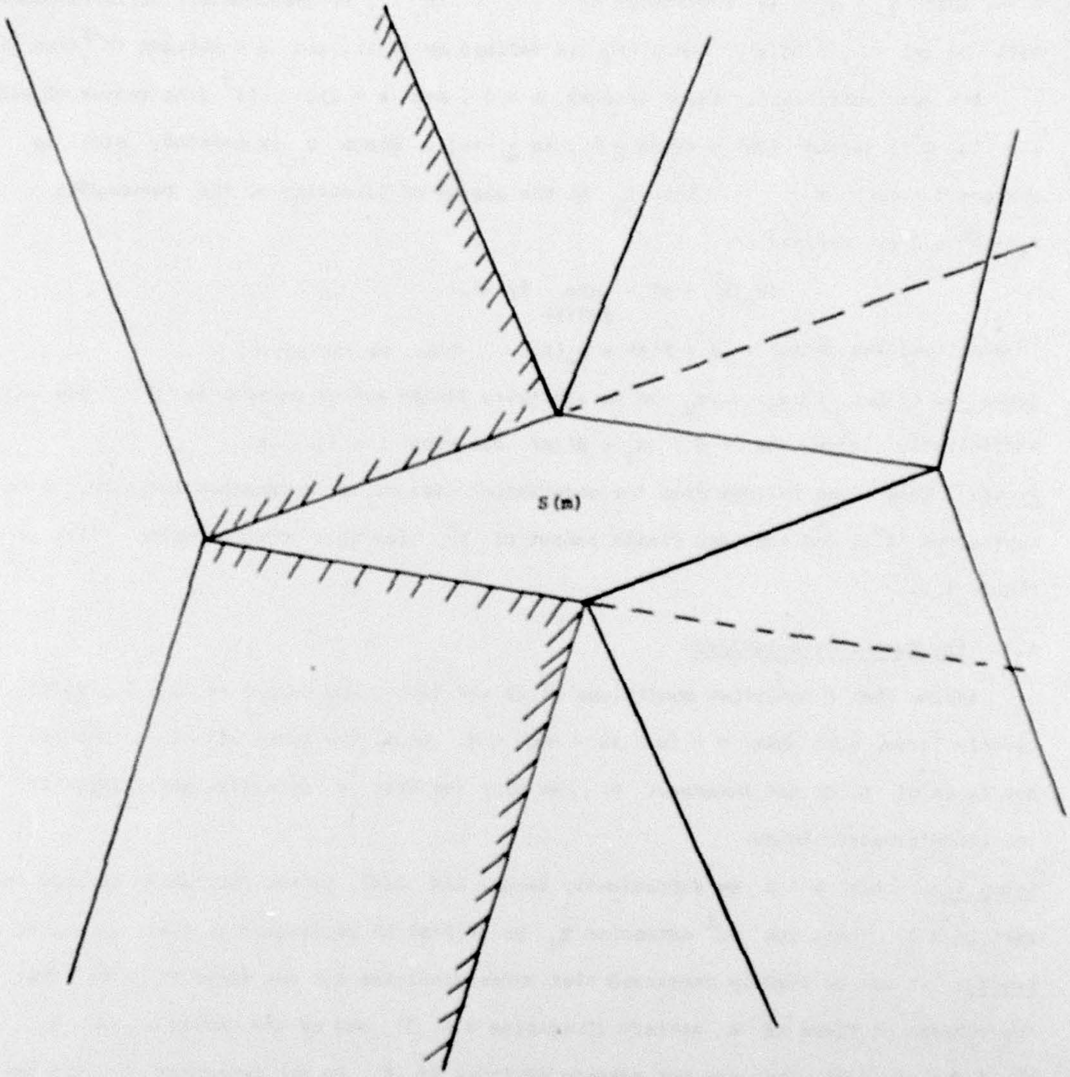

Figure 4.1

S'(m) is the shaded set. It is non-convex.

We can now prove our main theorem:

Theorem 4.4. Let f satisfy conditions (1.2) and (4.1). Then (1.1) has at most one solution for each q in R^n.

Proof: Let q be arbitrary, and let $S(q) = \{x : g(x) = q\}$. Now, let $x^1, x^2 \in S(q)$ with $x^1 \neq x^2$. From Lemma 4.1, for sufficiently large $m > 0$, x^1, x^2 are both members of $S'(m)$. But $g_m = g|S(m)$ is 1-1 on Σ_m, thus $g_m(x^1) \neq g_m(x^2)$. Since x^1, x^2 are in $S'(m)$, $g(x^1) = g_m(x^1)$ and $g(x^2) = g_m(x^2)$, we have a contradiction that to the fact that g_m is $1 - 1$. Thus our result follows.

We now show a condition that insures that (1.1) has a unique solution for each q in R^n.

Theorem 4.5. Let f satisfy the conditions of theorem 4.4. In addition, for each sequence $\{x_k\}_{k=1}^{\infty}$ such that $\|x_k\| \to \infty$, let $\|g(x^k)\| \to \infty$. Then (1.1) has a unique solution for each q in R^n.

Proof: Let x_0 be arbitrary in R^n, and let $q^* = f(x_0)$. Then $\#S(q) = 1$ (from Theorem 4.4, and the fact that $x_0 \in S(q^*)$). Hence $\deg(g, R^n, q^*) = +1$. Now, let q in R^n be arbitrary and consider the homotopy

$$H(x,t) = g(x) - (1-t)q^* - tq .$$

Since g is norm coercive, $H^{-1}(0)$ is bounded, and thus $\deg(g, R^n, q) = \deg(g, R^n, q^*) = +1$. Hence $S(q) \neq \phi$, and thus the result follows from Theorem 4.4.

§5. Appendix

Let $F : R^n \to R^n$ be a piecewise linear mapping on the subdivided polyhedron (R^n, Σ) as defined in section 2, i.e. the mapping $F|\sigma(I)$ is linear, so, for some $n \times n$ matrix A_I, $F|\sigma(I)(x) = A_I x$. We can then prove:

Lemma A1: There exist $n \times n$ matrices U and V such that

$$A_I^j = \begin{cases} V^j & j \in I \\ U^j & j \notin I \end{cases} \tag{A.1}$$

where A^j is the j^{th} column of the matrix A.

Proof: Define $U = A_\phi$ and $V = A_N$. Thus (1.1) holds for $I = \phi$ or N. Now, let $\phi \neq I \neq N$. Then, if u_j is the j^{th} unit vector in R^n,

$$-u_j \in \sigma(I) \cap \sigma(\phi) = \{x : x_i = 0 \text{ for } i \in I\} \qquad j \notin I$$

and

$$u_j \in \sigma(I) \cap \sigma(N) = \{x : x_i = 0 \text{ for } i \notin I\} \qquad j \in I.$$

Since F is continuous,

$$A_J u_j = A_\phi u_j \quad \text{for } j \notin I$$

and

$$A_J u_j = A_N u_j \quad \text{for } j \in I$$

and thus (A.1) and we are done.

We now prove our theorem

Theorem A.2: Let $F: R^n \to R^n$ and be piecewise linear on (R^n, Σ). Then, F maps R^n homeomorphically onto R^n if and only if $\det A_\phi \det A_I > 0$ for all $I \subseteq N$.

Proof: The necessity of the condition follows from the Theorem 2.3, Rhienboldt and Vandergraft [15]. We now show the sufficiency. Now F is a homeomorphism if and only if DF is, for any $n \times n$ nonsingular matrix D. Let $D = U^{-1}$, where U is defined in Lemma A.1. Also $\det U^{-1} A_I > 0$ for each I. As can be readily verified, the matrices I, $U^{-1}V$ and $U^{-1}A_I$ for $I \subseteq N$ satisfy the conditions of Theorem 6.1, Kojima and Saigal [7], and thus the sufficiency follows.

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In this paper we consider the problem of establishing the number of solutions to the complementarity problem. For the case when the Jacobian of the mapping has all principal minors negative, and satisfies a condition at infinity, we prove that the problem has either 0,1,2 or 3 solutions. We also show that when the Jacobian has all principal minors positive, and satisfies a condition at infinity, the problem has a unique solution.

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